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CONFIDENCE INTERVALS FOR A MEAN AND A PROPORTION IN THE 1/1

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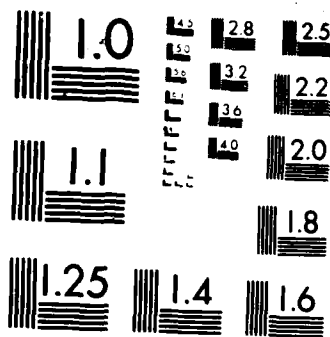
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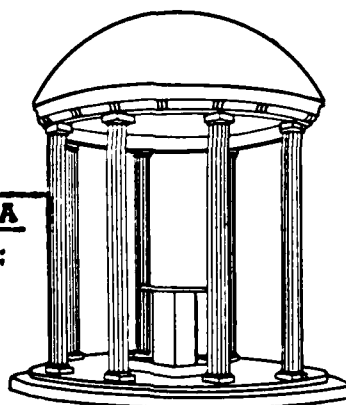
November 1986

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Curriculum in Operations Research and
Systems Analysis
University of North Carolina
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Abstract

This paper describes a $100 \times (1 - \alpha)$ confidence interval for the mean of a bounded random variable which is shorter than the interval that Chebyshev's inequality induces for small α and which avoids the error of approximation that assuming normality induces. The paper also presents an analogous development for deriving a $100 \times (1 - \alpha)$ confidence interval for a proportion.

Key words: Confidence interval, Proportion.

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Introduction

Let X_1, \dots, X_n be independent identically distributed random variables with $\mu = EX_1$, $\text{pr}(0 \leq X_1 \leq 1) = 1$ and $\bar{X}_n = (X_1 + \dots + X_n)/n$. This paper describes a method for deriving a $100 \times (1-\alpha)$ interval estimate of μ for finite n based on the probability inequality (Hoeffding 1963, Thm. 1, (2.1))

$$\text{pr}(\bar{X}_n - \mu \geq \epsilon) \leq e^{-nf(\epsilon, \mu)} \quad (1)$$

where

$$f(\epsilon, \mu) = (\epsilon + \mu)[\ln \mu - \ln(\mu + \epsilon)] + (1 - \epsilon - \mu)[\ln(1 - \mu) - \ln(1 - \mu - \epsilon)] \quad \epsilon < 1 - \mu \quad (2)$$

and

$$\lim_{\epsilon \rightarrow 1 - \mu} f(\epsilon, \mu) = \ln \mu.$$

To put our results in perspective, we first review the derivation of two commonly encountered confidence intervals. Let

$$k(x, \beta, m) = \{x + \beta^2/2m + \beta[\beta^2/4m^2 + x(1-x)/m]^{1/2}\} / (1 + \beta^2/m) \\ 0 \leq x \leq 1, \quad -\infty < \beta < \infty, \quad m = 1, 2, \dots \quad (3)$$

Then the interval $(k(\bar{X}_n, -\alpha^{-1/2}, n), k(\bar{X}_n, \alpha^{-1/2}, n))$ covers μ with confidence coefficient $> 1 - \alpha$, as a consequence of Chebyshev's inequality and the observation that $\text{var } X_1 = E(X_1 - \mu)^2 = E[X_1(X_1 - \mu)] \leq \mu(1 - \mu)$. Moreover, as a result of the central limit theorem, the interval $(k(\bar{X}_n, -\Phi^{-1}(1 - \alpha/2), n), k(\bar{X}_n, \Phi^{-1}(1 - \alpha/2), n))$ asymptotically ($n \rightarrow \infty$) covers μ with confidence coefficient $\geq 1 - \alpha$, where

$$\Phi^{-1}(\theta) = \{y: (2\pi)^{-1/2} \int_{-\infty}^y e^{-z^2/2} dz = \theta\}.$$

Although the Chebyshev interval holds for all n , the width of the resulting interval is considerably larger than that for the asymptotic normal one. For example $\alpha^{-1/2}/\phi^{-1}(1-\alpha/2) = 2.28$ for $\alpha=.05$. However, because of the nonuniform convergence of $(\bar{X}_n - \mu)/[\mu(1-\mu)/n]^{1/2}$, using the normal confidence interval obliges one to account for the inevitable error of approximation for finite n . This error makes difficult an assessment of whether or not the associated confidence coefficient truly exceeds $1-\alpha$, and can be especially bothersome in a Monte Carlo sampling experiment where the problem dictates the maximal interval width and the minimal acceptable confidence level. Even less appealing are interval estimates of the form $(\bar{X}_n - \beta(S_n^2/n)^{1/2}, \bar{X}_n + \beta(S_n^2/n)^{1/2})$ where

$$S_n^2 = (n-1)^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$$

and $\beta = \alpha^{-1/2}$ and $\beta = \phi^{-1}(1-\alpha/2)$ for the Chebyshev and normal cases respectively. Although intended to shorten the intervals by using the additional information in $S_n^2 \leq \bar{X}_n(1-\bar{X}_n)$ the substitution of S_n^2 for $\text{var } X_i$ induces an additional error of approximation in assessing whether or not the resulting confidence coefficient exceeds $1-\alpha$.

Hoeffding (1963) derived the probability inequality (1) for all bounded X_i . Previously Okamoto had derived (2) for X_i with the Bernoulli distribution, $\text{pr}(X_i=0) = 1-\mu$ and $\text{pr}(X_i=1) = \mu$, a result implicit in Chernoff (1952, Thm. 1 and Ex. 5). Theorem 1

provides the basis for constructing a confidence interval for μ based on Hoeffding's theorem.

Theorem 1. Let X_1, \dots, X_n be i.i.d. random variables with $\mu = EX_1$, $\text{pr}(0 \leq X_1 \leq 1) = 1$ and $\lambda = \max [\text{pr}(X_1=0), \text{pr}(X_1=1)]$. Then for $n \geq \ln(\alpha/2)/\ln \lambda$, $(\psi_1(\bar{X}_n, \alpha/2), \psi_2(\bar{X}_n, \alpha/2))$ covers μ with probability $> 1-\alpha$ where $\psi_1(\bar{X}_n, \alpha/2) \leq \bar{X}_n \leq \psi_2(\bar{X}_n, \alpha/2)$ are the solutions to

$$f(\bar{X}_n - \psi, \psi) = \frac{1}{n} \ln(\alpha/2). \quad (4)$$

Proof. Observe that

$$df(\epsilon, \mu)/d\epsilon = \ln[\mu(1-\mu-\epsilon)/(\mu+\epsilon)(1-\mu)] < 0 \quad 0 \leq \epsilon < 1-\mu,$$

$$f(0, \mu) = 0$$

and recall that

$$\lim_{\epsilon \rightarrow 1-\mu} f(\epsilon, \mu) = \ln \mu < 0.$$

Therefore, $e^{nf(\epsilon, \mu)}$ is monotone decreasing in ϵ with maximum 1 and minimum μ^n .

Consider the equal tail probability case and let

$$\begin{aligned} \epsilon(\mu, \alpha/2) &= \{\epsilon: e^{nf(\epsilon, \mu)} = \alpha/2\} & \text{if } \mu^n \leq \alpha/2 \\ &= 1-\mu & \text{if } \mu^n > \alpha/2. \end{aligned}$$

Then

$$\text{pr}[\bar{X}_n \geq h(\mu, \alpha/2)] \leq \alpha/2 \quad (5)$$

where

$$h(\mu, \alpha/2) = \mu + \varepsilon(\mu, \alpha/2).$$

For $\mu^n > \alpha/2$, $\text{pr}(\bar{X}_n \geq 1) \leq \lambda^n \leq \alpha/2$. For $\mu^n \leq \alpha/2$, we want to find the set of all μ 's satisfying (5). From $e^{nf(\varepsilon, \mu)} = \alpha/2$,

$$d\varepsilon(\mu, \alpha/2)/d\mu = -\{1 + \varepsilon/\mu(1-\mu) \ln[\mu(1-\mu-\varepsilon)/(1-\mu)(\mu+\varepsilon)]\} \quad (6)$$

so that

$$dh(\mu, \alpha/2)/d\mu > 0,$$

implying that h is monotone increasing in μ . Therefore, the set of μ 's of interest is $\{\mu: 0 < \mu \leq \psi_1(\bar{X}_n, \alpha/2)\}$ where

$$\psi_1(x, \alpha/2) = \{\psi: \psi + \varepsilon(\psi, \alpha/2) = x\},$$

which is precisely the solution to (4) in the interval $[0, \bar{X}_n]$. Consequently,

$$\text{pr}[\psi_1(\bar{X}_n, \alpha/2) \geq \mu] \leq \alpha/2$$

so that

$$\text{pr}[\psi_1(\bar{X}_n, \alpha/2) < \mu] > 1 - \alpha/2,$$

as required.

The upper bound $\psi_2(\bar{X}_n, \alpha/2)$ follows analogously, using

$$\text{pr}(-\bar{X}_n + \mu \geq \varepsilon) \leq e^{nf(\varepsilon, 1-\mu)}.$$

Observe that if X_i has a continuous distribution, $\lambda = 0$ and Theorem 1 holds for all sample sizes n . If $\text{pr}(a \leq X_i \leq b) = 1$, then Theorem 1 holds with $100 \times (1-\alpha)$ confidence interval $((b-a) \Psi_1((X_n-a)/(b-a), \alpha/2) + a, (b-a) \Psi_2((X_n-a)/(b-a), \alpha/2) + a)$. Although for Bernoulli data and small n , one can compute an exact confidence interval for μ , as in Blyth and Still (1983), this option loses its appeal as n increases and the potential for numerical error grows. Table 1 shows the lower bounds on n for $\alpha = .01$ and $.05$.

Insert Table 1 about here.

Using the dominant term (as $n \rightarrow \infty$) of the Taylor series of $f(x-\psi, \psi)$, one can readily show that as n increases

$$\Psi_2(\bar{X}_n, \alpha/2) - \Psi_1(\bar{X}_n, \alpha/2) = 2[2 \ln(2/\alpha) \bar{X}_n(1-\bar{X}_n)/n]^{1/2}. \quad (7)$$

To order $n^{-1/2}$, the Chebyshev and normal intervals have widths $2[\alpha^{-1} \bar{X}_n(1-\bar{X}_n)/n]^{1/2}$ and $2\Phi^{-1}(1-\alpha/2)[\bar{X}_n(1-\bar{X}_n)/n]^{1/2}$ respectively. Table 2 compares these widths for $\alpha = .01$ and $.05$.

Confidence Interval for a Proportion

Let $(X_1, Y_1), \dots, (X_n, Y_n)$ denote i.i.d. random vectors with $\mu_X = EX_1$, $\mu_Y = EY_1$, $\text{pr}(0 \leq X_1 \leq 1) = 1$, $\text{pr}(0 \leq Y_1 \leq 1) = 1$, $\text{pr}(Y_1 \leq X_1) = 1$, $\phi = \mu_Y/\mu_X$, $\bar{X}_n = (X_1 + \dots + X_n)/n$ and $\bar{Y}_n = (Y_1 + \dots + Y_n)/n$. Also, let

$$r(x, y, \beta, m) = \{xy + \beta^2/m + \beta[\beta^2/4m^2 + y(x-y)/m]^{1/2}\} / (x^2 + \beta^2/m) \\ 0 \leq y \leq x \leq 1, \quad -\infty < \beta < \infty, \quad m=1, 2, \dots \quad (8)$$

Then $(r(\bar{X}_n, \bar{Y}_n, -\beta, n), r(\bar{X}_n, \bar{Y}_n, \beta, n))$ with $\beta = \alpha^{-1/2}$ covers ϕ with confidence coefficient $> 1-\alpha$, and with $\beta = \phi^{-1}(1-\alpha/2)$ asymptotically $(n \rightarrow \infty)$ covers ϕ with confidence coefficient $1-\alpha$. These results again follow from Chebyshev's inequality, the central limit theorem and the observation that

$$\text{var}(Y_i - \phi X_i) = \text{var}(Y_i - \phi X_i + \phi) \leq \phi(1-\phi).$$

Again, one can derive a $100 \times (1-\alpha)$ confidence interval, shorter than the one that Chebyshev's inequality offers for small α and that avoids the error of approximation that assuming normality induces. Let

$$W_i = Y_i - \phi X_i + \phi$$

so that $\phi = EW_i$ and $\text{pr}(0 \leq W_i \leq 1) = 1$. Then for $\bar{W}_n = (W_1 + \dots + W_n)/n$, (1) applies in the form

$$\text{pr}(\bar{Y}_n - \phi \bar{X}_n \geq \epsilon) = \text{pr}(\bar{W}_n - \phi \geq \epsilon) \leq e^{nf(\epsilon, \phi)}. \quad (9)$$

This establishes the basis for Theorem 2.

Theorem 2. Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be i.i.d. random vectors with $\mu_X = EX_1$, $\mu_Y = EY_1$, $\text{pr}(0 \leq X_1 \leq 1) = 1$, $\text{pr}(0 \leq Y_1 \leq 1) = 1$, $\text{pr}(Y_1 \leq X_1) = 1$, $\lambda = \max[\text{pr}(Y_1=0), \text{pr}(Y_1=1)]$, $\bar{X}_n = (X_1 + \dots + X_n)/n$, $\bar{Y}_n = (Y_1 + \dots + Y_n)/n$ and $\phi = \mu_Y/\mu_X$. Then for $n \geq \ln(\alpha/2)/\ln \lambda$, $(\gamma_1(\bar{X}_n, \bar{Y}_n, \alpha/2), \gamma_2(\bar{X}_n, \bar{Y}_n, \alpha/2))$ covers ϕ with probability $> 1-\alpha$

where $\gamma_1(x, y, \theta) \leq y/x \leq \gamma_2(x, y, \theta)$, are the solutions of

$$f(y-\gamma x, \gamma) = \frac{1}{n} \ln \theta \quad 0 \leq y \leq x \leq 1, \quad (10)$$

f being defined in (2).

Proof. Let

$$\begin{aligned} \varepsilon(\phi, \alpha/2) &= \{ \varepsilon : f(\varepsilon, \phi) = \frac{1}{n} \ln(\alpha/2) \} && \text{if } \phi^n \leq \alpha/2 \\ &= 1 - \phi && \text{if } \phi^n > \alpha/2. \end{aligned}$$

Then

$$\text{pr}[\bar{Y}_n \geq g(\phi, \alpha/2, \bar{X}_n)] \leq \alpha/2 \quad (11)$$

where

$$g(\phi, \alpha/2, x) = \phi x + \varepsilon(\phi, \alpha/2).$$

For $\phi^n > \alpha/2$,

$$\begin{aligned} \text{pr}(\bar{Y}_n - \phi \bar{X}_n \geq 1 - \phi) &= \text{pr}(\bar{Y}_n = 1, \bar{X}_n = 1) \\ &= \text{pr}(\bar{X}_n = 1 | \bar{Y}_n = 1) \text{pr}(\bar{Y}_n = 1) \\ &= \text{pr}(\bar{Y}_n = 1) \\ &= \lambda^n \leq \alpha/2. \end{aligned}$$

For $\phi^n \leq \alpha/2$, we want to find the set of all ϕ 's satisfying (11). Observe that

$$dg(\phi, \alpha/2, x)/d\phi = x + \partial \varepsilon(\phi, \alpha/2)/\partial \phi$$

where (6) gives $\partial \varepsilon(\phi, \alpha/2)/\partial \phi$. Using the inequalities $z/(1+z) < \ln(1+z) < z$ for $z > -1$ and $z \neq 0$, one has

$$\partial g(\phi, \alpha/2, \bar{y}) / \partial \phi \geq -\phi \epsilon / (1-\phi)(1-\epsilon).$$

Since $\epsilon > 0$, $\phi \leq \bar{Y}_n / \bar{X}_n$ so that

$$-\phi / (1-\phi) \geq (\epsilon - \bar{Y}_n) \epsilon / (1-\epsilon)(\bar{X}_n - \bar{Y}_n + \epsilon) > (\epsilon - \bar{Y}_n) / (1-\epsilon),$$

and finally

$$\begin{aligned} \partial g(\phi, \alpha/2, \bar{X}_n) / \partial \phi &\geq \bar{X}_n + (\epsilon - \bar{Y}_n) / (1-\epsilon) = [\bar{X}_n - \bar{Y}_n + \epsilon(1 - \bar{X}_n)] / (1-\epsilon) \\ &> 0. \end{aligned}$$

Therefore, the set of ϕ 's of interest is $\{\phi: 0 < \phi \leq \gamma_1(\bar{X}_n, \bar{Y}_n, \alpha/2)\}$

where

$$\gamma_1(x, y, \theta) = \{\psi: \psi x + \epsilon(\psi, \alpha/2) = y, \quad 0 \leq y \leq x \leq 1, \quad 0 < \theta < 1\},$$

which is precisely the solution to (10). Consequently,

$$\text{pr}[\gamma_1(\bar{X}_n, \bar{Y}_n, \alpha/2) \geq \phi] \leq \alpha/2$$

so that

$$\text{pr}[\phi > \gamma_1(\bar{X}_n, \bar{Y}_n, \alpha/2)] > 1 - \alpha/2,$$

as required. The upper bound $\gamma_2(\bar{X}_n, \bar{Y}_n, \alpha/2)$ follows analogously using

$$\text{pr}(-\bar{W}_n + \phi \geq \epsilon) \leq e^{-nf(\epsilon, 1-\phi)}.$$

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Table 1

$$n_0 = \min \{n: \lambda^n \leq \alpha/2\}$$

α	λ	n_0	α	λ	n_0
.01	.90	51	.05	.90	36
	.99	528		.99	368
	.999	5296		.999	3688
	.9999	52981		.9999	36887

Table 2

Comparison of Interval Widths

α	$\alpha^{-1/2}/\phi^{-1}(1-\alpha/2)$	$\alpha^{-1/2}/[2\ln(2/\alpha)]^{1/2}$	$[2\ln(2/\alpha)]^{1/2}/\phi^{-1}(1-\alpha/2)$
.01	3.88	3.07	1.26
.05	2.28	1.65	1.39

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